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On the eigenfunctions of the Dirac operator on spheres and real hyperbolic spaces

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Abstract

The eigenfunctions of the Dirac operator on spheres and real hyperbolic spaces of arbitrary dimension are computed by separating variables in geodesic polar coordinates. These eigenfunctions are then used to derive the heat kernel of the iterated Dirac operator on these spaces. They are then studied as cross sections of homogeneous vector bundles, and a group-theoretic derivation of the spinor spherical functions and heat kernel is given based on Harish–Chandra's formula for the radial part of the Casimir operator.

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1. Introduction

The N-dimensional sphere (S^N) and the real hyperbolic space (H^N), which are "dual" to each other as symmetric spaces [10], are maximally symmetric. This high degree of symmetry allows one to compute explicitly the eigenfunctions of the Laplacian for various fields on these spaces. These eigenfunctions can be used in studying field theory in de Sitter and anti-de Sitter space–times since S^4 and H^4 are Euclidean sections of these space–times. Also S^3 and H^3 appear as the spatial sections of cosmological models, and various field equations and their solutions on these spaces have physical applications in this context. In addition to these applications, fields on S^N and H^N provide concrete examples for the theory

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of homogeneous vector bundles, and consequences of various theorems can explicitly be worked out.

Recently the authors presented the eigenfunctions of the Laplacian for the transverse-traceless totally symmetric tensor fields [4] and for the totally antisymmetric tensor fields (p-forms) [5]. These fields were also analyzed in the light of group theory using the fact that they are cross sections of homogeneous vector bundles. As a continuation of these works we study in this paper the spinor fields satisfying the Dirac equation $\nabla\!\!\!/ \psi = \mathrm{i}\lambda\psi$ and the heat kernel for $\nabla\!\!\!/^2$ on S^N and H^N. The paper is organized as follows. In Section 2 the appropriately normalized eigenfunctions of the Dirac operator on S^N with arbitrary N are presented using geodesic polar coordinates. The solutions on S^N are expressed in terms of those on S^{N-1}. Then we derive the degeneracies of the Dirac operator using the spinor eigenfunctions. Next, spinor eigenfunctions on H^N are obtained by analytically continuing those on S^N. Then they are used to derive the spinor spectral function (Plancherel measure) on H^N. In Section 3 the results of Section 2 are used to write down the heat kernel for the iterated Dirac operator $\nabla\!\!\!/^2$ on these spaces.

Section 4 is devoted to a group-theoretic analysis of spinor fields on S^N and H^N. We use the fact that spinor fields on these symmetric spaces are cross sections of the homogeneous vector bundles associated to the fundamental spinor representation(s) of $Spin(N)$. By applying harmonic analysis for homogeneous vector bundles to $S^N = Spin(N+1)/Spin(N)$, in combination with the formula for the radial part of the Casimir operator given by Harish–Chandra, we derive the spinor spherical functions and the heat kernel of the iterated Dirac operator on S^N. Then we apply harmonic analysis for homogeneous vector bundles to $H^N = Spin(N,1)/Spin(N)$, and rederive some results of Section 3 for this space.

2. Spherical modes of the Dirac operator

2.1. Spin structures and spinor fields

A spin structure on an oriented n-dimensional Riemannian manifold M^n is a principal bundle over M^n with structure group $Spin(n)$ (denoted $Spin(M^n)$ and sometimes called the bundle of spinor frames) together with a two-to-one bundle homomorphism f of $Spin(M^n)$ onto $SO(M^n)$ which is fiber preserving and group equivariant with respect to the standard two-to-one covering homomorphism (see, e.g., [13]). Let τ be the (unique) fundamental spinor representation of $Spin(n)$ (of dimension $2^{(n-1)/2}$) for n odd, and $\tau_+ \oplus \tau_-$, where $\tau_\pm$ are the two fundamental spinor representations of $Spin(n)$ (of dimension $2^{(n/2)-1}$ each), for n even. We define a Dirac spinor field on M^n to be a cross section of the vector bundle E^τ over M^n with typical fibre $\mathbb{C}^{2^{[n/2]}}$ and structure group $Spin(n)$ associated with $Spin(M^n)$ via the representation τ (see [7, p.418], for the definition of a general spinor field). In this paper we consider only Dirac spinors, and from now on the word "spinor" means Dirac spinor. Among the spaces we consider, i.e. S^N and H^N, only S^1 admits more than one spin structure. There are two inequivalent spin structures on S^1, and we choose the nontrivial one (see [8,15]) for the reason which will become clear later.

For explicit computations, such as the ones performed below, we need to express a spinor field and its covariant derivative with respect to a (local) moving orthonormal frame $\{\mathbf{e}_a\}_{a=1,\ldots,n}$ on M^n in components. A metric connection $\boldsymbol{\omega}$ induces a spin connection on M^n (i.e. a connection on $Spin(M^n)$) and a covariant derivative for the cross sections. Given a spinor ψ, the components of the spinor $\nabla_{\mathbf{e}_a}\psi$ in a local spinor frame $\{\theta_A\}_{A=1,\ldots,2^{[n/2]}}$ (i.e. a set of linearly independent local cross sections of E^τ) are given by (see [7, p.420])

$$(\nabla_{\mathbf{e}_a}\psi)^A = \mathbf{e}_a(\psi^A) - \tfrac{1}{2}\omega_{abc}(\Sigma^{bc})^A{}_B\psi^B, \tag{2.1}$$

where ω_{abc} are the connection coefficients in the frame $\mathbf{e}_a$ [defined by $\nabla_{\mathbf{e}_a}\mathbf{e}_b = \omega_{abc}\mathbf{e}_c$], and summation over repeated indices is understood from now on. The matrices Σ^{ab} are the generators of the representation τ of $Spin(n)$. They can be expressed as $\Sigma^{ab} = \frac{1}{4}[\Gamma^a, \Gamma^b]$, where $\Gamma^a = (\Gamma^a)^A{}_B$ ($a = 1, \ldots, n$, $A, B = 1, \ldots, 2^{[n/2]}$) are Dirac matrices satisfying

$$\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = 2\delta^{ab}\mathbf{1}, \quad a, b = 1, \ldots, n, \tag{2.2}$$

where $\mathbf{1}$ is the unit matrix and δ^{ab} is the Kronecker symbol. The Dirac operator is then defined by

$$\nabla\!\!\!/\,\psi = \Gamma^a\nabla_{\mathbf{e}_a}\psi. \tag{2.3}$$

2.2. S^N

The metric on S^N may be written in geodesic polar coordinates as

$$\mathrm{d}s_N^2 = \mathrm{d}\theta^2 + f^2(\theta)\,\mathrm{d}s_{N-1}^2 = \mathrm{d}\theta^2 + f^2(\theta)\sum_{i,j=1}^{N-1}\tilde{g}_{ij}\,\mathrm{d}\omega^i\otimes\mathrm{d}\omega^j, \tag{2.4}$$

where θ is the geodesic distance from the origin (north-pole), $f(\theta) = \sin\theta$, and $\{\omega^i\}$ are coordinates on S^{N-1}, with metric tensor $\tilde{g}_{ij}(\omega) = \langle\partial/\partial\omega^i, \partial/\partial\omega^j\rangle$. Let $\{\tilde{\mathbf{e}}_j\}$ be a vielbein on S^{N-1}, with anholonomy and (Levi-Civita) connection coefficients

$$[\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_j] = \sum_{k=1}^{N-1}\tilde{C}_{ijk}\tilde{\mathbf{e}}_k, \tag{2.5}$$

$$\tilde{\omega}_{ijk} = \langle\tilde{\nabla}_{\tilde{\mathbf{e}}_i}\tilde{\mathbf{e}}_j, \tilde{\mathbf{e}}_k\rangle = \tfrac{1}{2}(\tilde{C}_{ijk} - \tilde{C}_{ikj} - \tilde{C}_{jki}). \tag{2.6}$$

We shall work in the geodesic polar coordinates vielbein $\{\mathbf{e}_a\}_{a=1,\ldots,N}$ on S^N defined by

$$\mathbf{e}_N = \partial_\theta \equiv \partial/\partial\theta, \qquad \mathbf{e}_j = \frac{1}{f(\theta)}\tilde{\mathbf{e}}_j, \quad j = 1, \ldots, N-1. \tag{2.7}$$

The only nonvanishing components of the Levi-Civita connection ω_{abc} in the frame $\{\mathbf{e}_a\}$ are found to be

$$\omega_{ijk} = (1/f)\tilde{\omega}_{ijk}, \qquad \omega_{iNk} = -\omega_{ikN} = (f'/f)\delta_{ik}, \quad i, j, k = 1, \ldots, N-1, \tag{2.8}$$

where a prime denotes differentiation with respect to the argument. We use the spin connection on S^N induced uniquely by the Levi-Civita connection ω. We shall now solve for the eigenfunctions of $\nabla\!\!\!\!/$ on S^N by separating variables in geodesic polar coordinates.

Case 1: *N even*. We choose the Γ-matrices such that

$$\Gamma^N = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \Gamma^j = \begin{pmatrix} 0 & \mathrm{i}\tilde{\Gamma}^j \\ -\mathrm{i}\tilde{\Gamma}^j & 0 \end{pmatrix}, \quad j = 1, \ldots, N-1, \tag{2.9}$$

where $\mathrm{i} = \sqrt{-1}$ and $\{\tilde{\Gamma}^j, \tilde{\Gamma}^k\} = 2\delta^{jk}\mathbf{1}$. Using Eqs. (2.8) and (2.3), it is straightforward to derive the following expression for the Dirac operator in the vielbein (2.7):

$$\nabla\!\!\!\!/\,\psi = \left(\partial_\theta + \frac{N-1}{2}\frac{f'}{f}\right)\Gamma^N \psi + \frac{1}{f}\begin{pmatrix} 0 & \mathrm{i}\,\tilde{\nabla\!\!\!\!/} \\ -\mathrm{i}\,\tilde{\nabla\!\!\!\!/} & 0 \end{pmatrix}\psi, \tag{2.10}$$

where $\tilde{\nabla\!\!\!\!/} = \tilde{\Gamma}^j \tilde{\nabla}_{\tilde{\mathbf{e}}_j}$ is the Dirac operator on S^{N-1}. Now suppose that we have solved the eigenvalue equation on S^{N-1}, i.e.

$$\tilde{\nabla\!\!\!\!/}\,\chi_{lm}^{(\pm)} = \pm\mathrm{i}(l+\rho)\chi_{lm}^{(\pm)}. \tag{2.11}$$

Here, the index $l = 0, 1, \ldots,$ labels the eigenvalues of the Dirac operator on the $(N-1)$-sphere, and $\rho \equiv \frac{1}{2}(N-1)$, [1] while the index m runs from 1 to the degeneracy d_l. Define

$$\psi \equiv \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}. \tag{2.12}$$

Since the dimension of χ is the same as the dimension of $\phi_\pm$, i.e. $2^{(N/2)-1}$, one can separate variables in the following way:

$${}^{(1)}\phi_{+nlm}(\theta, \Omega) = \phi_{nl}(\theta)\chi_{lm}^{(-)}(\Omega), \tag{2.13}$$

$${}^{(2)}\phi_{+nlm}(\theta, \Omega) = \psi_{nl}(\theta)\chi_{lm}^{(+)}(\Omega), \tag{2.14}$$

and similarly for the "lower" spinor ϕ_-. Here $\Omega \in S^{N-1}$, and $n = 0, 1, \ldots,$ labels the eigenvalues $-\lambda_{n,N}^2$ of $\nabla\!\!\!\!/^2$ on S^N and $n \geq l$ (see later).

Substituting (2.13) in the equation $\nabla\!\!\!\!/^2\psi = -\lambda_{n,N}^2\psi$, we obtain the following equation for the scalar functions ϕ_{nl}:

$$\left[\left(\frac{\partial}{\partial\theta} + \rho\cot\theta\right)^2 - \frac{(l+\rho)^2}{\sin^2\theta} + (l+\rho)\frac{\cos\theta}{\sin^2\theta}\right]\phi_{nl} = -\lambda_{n,N}^2\,\phi_{nl}. \tag{2.15}$$

The unique regular solution to this equation is given in terms of Jacobi polynomials $P_n^{(a,b)}(x)$ [9], up to a normalization factor,

$$\phi_{nl}(\theta) = (\cos\tfrac{1}{2}\theta)^{l+1}(\sin\tfrac{1}{2}\theta)^l\, P_{n-l}^{((N/2)+l-1,\,(N/2)+l)}(\cos\theta), \tag{2.16}$$

[1] The spectrum of the Dirac operator on spheres is well known (see, e.g. [15]). In fact our procedure gives an independent proof of (2.17) by induction over the dimension N. Eq. (2.11) may then be assumed as the inductive hypothesis in this proof.

with $n - l \geq 0$ – this condition is needed for the regularity of the eigenfunctions – and with the eigenvalues

$$\lambda_{n,N}^2 = (n + \tfrac{1}{2}N)^2. \tag{2.17}$$

By proceeding in a similar way with the functions ψ_{nl} in (2.14) we find

$$\psi_{nl}(\theta) = (\cos\tfrac{1}{2}\theta)^l (\sin\tfrac{1}{2}\theta)^{l+1} P_{n-l}^{((N/2)+l,\,(N/2)+l-1)}(\cos\theta) \tag{2.18}$$

$$= (-1)^{n-l}\phi_{nl}(\pi - \theta). \tag{2.19}$$

Define

$$\psi_{\pm nlm}^{(-)}(\theta, \Omega) = \frac{c_N(nl)}{\sqrt{2}} \begin{pmatrix} \phi_{nl}(\theta)\chi_{lm}^{(-)}(\Omega) \\ \pm \mathrm{i}\psi_{nl}(\theta)\chi_{lm}^{(-)}(\Omega) \end{pmatrix}, \tag{2.20}$$

$$\psi_{\pm nlm}^{(+)}(\theta, \Omega) = \frac{c_N(nl)}{\sqrt{2}} \begin{pmatrix} \mathrm{i}\psi_{nl}(\theta)\chi_{lm}^{(+)}(\Omega) \\ \pm \phi_{nl}(\theta)\chi_{lm}^{(+)}(\Omega) \end{pmatrix}. \tag{2.21}$$

Then we find that these spinors satisfy the first-order Dirac equation

$$\nabla\!\!\!/\, \psi_{\pm nlm}^{(s)} = \pm \mathrm{i}(n + \tfrac{1}{2}N)\psi_{\pm nlm}^{(s)}, \tag{2.22}$$

where $s = \pm$. We require them to satisfy the normalization condition,

$$\langle \psi_{+nlm}^{(s)}, \psi_{+n'l'm'}^{(s')} \rangle \equiv \int_{S^N} \mathrm{d}\Omega_N \psi_{+nlm}^{(s)}(\theta, \Omega)^\dagger \psi_{+n'l'm'}^{(s')}(\theta, \Omega) = \delta_{nn'}\delta_{ll'}\delta_{mm'}\delta_{ss'}, \tag{2.23}$$

with an analogous relation for $\psi_+^{(s)} \to \psi_-^{(s)}$. We note that $\langle \psi_{+nlm}^{(s)}, \psi_{-n'l'm'}^{(s')} \rangle = 0$ for any choice of indices. Suppose that the spinors $\chi_{lm}^{(\pm)}(\Omega)$ are normalized by

$$\int_{S^{N-1}} \mathrm{d}\Omega_{N-1} \chi_{lm}^{(s)}(\Omega)^\dagger \chi_{l'm'}^{(s')}(\Omega) = \delta_{ll'}\delta_{mm'}\delta_{ss'}. \tag{2.24}$$

Then the normalization factor $c_N(nl)$ can be determined as

$$|c_N(nl)|^{-2} = \frac{2^{N-2}\left|\Gamma(\frac{1}{2}N + n)\right|^2}{(n-l)!(N+n+l-1)!}. \tag{2.25}$$

The degeneracy for the eigenvalue $+\mathrm{i}(n + \frac{1}{2}N)$ [or $-\mathrm{i}(n + \frac{1}{2}N)$] is given by letting $n = n'$, $l = l'$, $m = m'$, $s = s'$ in (2.23) and summing over l, m, s as

$$D_N(n) = \int_{S^N} \mathrm{d}\Omega_N \sum_{slm} \psi_{+nlm}^{(s)}(\theta, \Omega)^\dagger \psi_{+nlm}^{(s)}(\theta, \Omega). \tag{2.26}$$

We use the fact that the sum over s, l, m inside the integral is constant over S^N, so that it may be calculated for $\theta \to 0$, where only the $l = 0$ term survives. We also note that the volume of S^N is

$$\Omega_N = \frac{2\pi^{(N+1)/2}}{\Gamma(\frac{1}{2}(N+1))}. \tag{2.27}$$

Now the degeneracy d_0 of $+i\rho$ (or $-i\rho$) on S^{N-1} (cf. (2.11)) coincides with the dimension $2^{(N/2)-1}$ of the fundamental spinor representations $\tau_\pm$ of $Spin(N)$. Thus $m = 1, 2, \ldots, 2^{(N/2)-1}$ for $l = 0$ in (2.26), and we have the identity

$$2^{(N/2)-1} = \Omega_{N-1} \sum_m \chi_{0m}^{(s)}(\Omega)^\dagger \chi_{0m}^{(s)}(\Omega). \tag{2.28}$$

From this equation and Eqs. (2.26) and (2.27) we obtain

$$D_N(n) = \frac{2^{N/2}(N+n-1)!}{n!(N-1)!}. \tag{2.29}$$

Case 2: *N odd* (≥ 3). In this case a Dirac spinor on S^N is irreducible under $Spin(N)$ and the dimension of the Γ-matrices is $2^{(N-1)/2}$, the same as on S^{N-1}. Thus for $a = 1, \ldots, N-1$ we choose Γ^a as before, and for $a = N$ we choose

$$\Gamma^N = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} = (-i)^{(N-1)/2}\Gamma^1\Gamma^2\cdots\Gamma^{N-1}. \tag{2.30}$$

The Dirac operator in the geodesic polar coordinates vielbein (2.7) takes the form

$$\nabla\!\!\!\!/\,\psi = (\partial_\theta + \rho\cot\theta)\Gamma^N\psi + (1/\sin\theta)\,\tilde{\nabla\!\!\!\!/}\,\psi. \tag{2.31}$$

Now, suppose that $\chi_{lm}^{(-)}$ satisfy

$$\tilde{\nabla\!\!\!\!/}\,\chi_{lm}^{(-)} = -i(l+\rho)\chi_{lm}^{(-)}, \quad l = 0, 1, \ldots \tag{2.32}$$

Then $\chi_{lm}^{(+)} \equiv \Gamma^N \chi_{lm}^{(-)}$ is the eigenfunction of $\tilde{\nabla\!\!\!\!/}$ with eigenvalue $+i(l+\rho)$. Define $\hat{\chi}_{lm}^{(\pm)}$ by

$$\hat{\chi}_{lm}^{(-)} = (1/\sqrt{2})(1 + i\Gamma^N)\chi_{lm}^{(-)}, \tag{2.33}$$
$$\hat{\chi}_{lm}^{(+)} = \Gamma^N \hat{\chi}_{lm}^{(-)}. \tag{2.34}$$

The normalized eigenfunctions of the first-order Dirac operator are found to be

$$\psi_{\pm nlm}(\theta, \Omega) = (c_N(nl)/\sqrt{2})\left[\phi_{nl}(\theta)\hat{\chi}_{lm}^{(-)}(\Omega) \pm i\psi_{nl}(\theta)\hat{\chi}_{lm}^{(+)}(\Omega)\right], \tag{2.35}$$

where

$$\nabla\!\!\!\!/\,\psi_{\pm nlm}(\theta, \Omega) = \pm i(n + \tfrac{1}{2}N)\psi_{\pm nlm}(\theta, \Omega). \tag{2.36}$$

The computation of the dimensionality is similar to that for N even. We find

$$D_N(n) = \frac{2^{(N-1)/2}(N+n-1)!}{n!(N-1)!}. \tag{2.37}$$

We have shown how to separate variables in the Dirac equation written in geodesic polar coordinates. In fact we have set up an induction procedure by which the spinor modes on S^N can be recursively calculated starting from the spinor modes on S^2. For $N = 2$, writing $ds_2^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$, one simply has $\tilde{\nabla\!\!\!\!/} = \partial/\partial\varphi$. We need to find which eigenvalues of this operator are allowed. Spinors on S^2 should transform as a double-valued spin-$\frac{1}{2}$

representation of $SO(2)$ under the rotation of zweibein. Now, the loop defined by $0 \leq \varphi < 2\pi$ with $\theta = \text{const.}$ in the bundle of frames over S^2 is homotope to the 2π-rotation at a point for our system of zweibein. Hence, the spinor field must change sign when it goes around this loop. Thus, we have $\partial/\partial\varphi = \pm\frac{1}{2}\mathrm{i}, \pm\frac{3}{2}\mathrm{i}, \ldots$ [2] This amounts to using the nontrivial spin structure on S^1 (see [15]).

2.3. H^N

The real hyperbolic space H^N is the noncompact partner of S^N. The metric on H^N in geodesic polar coordinates takes the form (2.4) with $\theta \to y$ and $f(y) = \sinh y$, where y is the geodesic distance from the origin. By repeating the same steps as on S^N one obtains a hypergeometric equation for the spherical modes $\phi_{\lambda l}(y)$. The spectrum of $\nabla\!\!\!\!/^2$ is given by $-\lambda^2$ where λ is now a real and continuous label. The final results on H^N and S^N are related by analytic continuation in the geodesic distance. More precisely, one expresses the Jacobi polynomials in terms of the hypergeometric function and makes the replacements

$$\theta \to \mathrm{i}y, \qquad n \to -\mathrm{i}\lambda - \tfrac{1}{2}N \tag{2.38}$$

in the spinor modes found above on S^N. The result is as follows. Define

$$\phi_{\lambda l}(y) = (\cosh\tfrac{1}{2}y)^{l+1}(\sinh\tfrac{1}{2}y)^{l}F(\tfrac{1}{2}N+l+\mathrm{i}\lambda, \tfrac{1}{2}N+l-\mathrm{i}\lambda, \tfrac{1}{2}N+l, -\sinh^2\tfrac{1}{2}y), \tag{2.39}$$

$$\begin{aligned}\psi_{\lambda l}(y) = {} & \frac{2\lambda}{N+2l}(\cosh\tfrac{1}{2}y)^{l}(\sinh\tfrac{1}{2}y)^{l+1} \\ & \times F(\tfrac{1}{2}N+l+\mathrm{i}\lambda, \tfrac{1}{2}N+l-\mathrm{i}\lambda, \tfrac{1}{2}N+l+1, -\sinh^2\tfrac{1}{2}y),\end{aligned} \tag{2.40}$$

where $\lambda \in \mathbb{R}^+$, and $l = 0, 1, \ldots, \infty$. For N even we have

$$\psi^{(-)}_{\pm\lambda lm}(y, \Omega) \equiv \frac{c_N(\lambda l)}{\sqrt{2}}\begin{pmatrix}\phi_{\lambda l}(y)\chi^{(-)}_{lm}(\Omega) \\ \pm\mathrm{i}\psi_{\lambda l}(y)\chi^{(-)}_{lm}(\Omega)\end{pmatrix}, \tag{2.41}$$

where

$$\nabla\!\!\!\!/\,\psi^{(-)}_{\pm\lambda lm} = \pm\mathrm{i}\lambda\psi^{(-)}_{\pm\lambda lm}. \tag{2.42}$$

The modes $\psi^{(+)}_{\pm\lambda lm}$ are obtained by interchanging $\phi_{\lambda l}(y)$ and $\mathrm{i}\psi_{\lambda l}(y)$, and letting $\chi^{(-)}_{lm} \to \chi^{(+)}_{lm}$ on the right-hand side. The (continuous-spectrum) normalization constant $c_N(\lambda l)$ is determined from the condition

$$\begin{aligned}&\int_0^{+\infty}\int_{S^{N-1}} \psi^{(s)}_{\sigma\lambda lm}(y,\Omega)^{\dagger}\psi^{(s')}_{\sigma'\lambda' l'm'}(y,\Omega)(\sinh y)^{N-1}\,\mathrm{d}y\,\mathrm{d}\Omega_{N-1} \\ &\quad = \delta(\lambda-\lambda')\delta_{ll'}\delta_{mm'}\delta_{ss'}\delta_{\sigma\sigma'},\end{aligned} \tag{2.43}$$

[2] It also follows that the spinor must vanish both at $\theta = 0$ and π if $\partial/\partial\varphi \neq \pm\frac{1}{2}\mathrm{i}$. This is automatically satisfied by requiring regularity of the solutions.

and can be found with the method used for scalar and tensor fields [4] as

$$|c_N(\lambda l)|^2 = c_N |c_l(\lambda)|^{-2}, \tag{2.44}$$

where $c_N = 2^{N-2}/\pi$ and

$$c_l(\lambda) = \frac{2^{N-2}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}N + l)\Gamma(\frac{1}{2} + \mathrm{i}\lambda)}{\Gamma(\frac{1}{2}N + l + \mathrm{i}\lambda)}. \tag{2.45}$$

We define the spectral function $\mu(\lambda)$, which is needed to find the heat kernel, by

$$\mu(\lambda) \equiv \frac{\Omega_{N-1}}{c_N g(\frac{1}{2})} \sum_{slm} \psi^{(s)}_{+\lambda lm}(0)^\dagger \psi^{(s)}_{+\lambda lm}(0), \tag{2.46}$$

where again $s = \pm$ and the spin factor $g(\frac{1}{2}) = 2^{N/2}$ is the dimension of the spinor. Then we find

$$\mu(\lambda) = |c_0(\lambda)|^{-2} = \frac{\pi}{2^{2N-4}} \left| \frac{\Gamma(\frac{1}{2}N + \mathrm{i}\lambda)}{\Gamma(\frac{1}{2}N)\Gamma(\frac{1}{2} + \mathrm{i}\lambda)} \right|^2 . \tag{2.47}$$

The computation proceeds almost in the same way for N odd. The modes are obtained from (2.35) by the substitution (2.38) and the formula for $\mu(\lambda)$ is again (2.47).

3. The spinor heat kernel

3.1. The compact case

In terms of a local moving frame $\{\mathbf{e}_a\}$ on S^N and a local spinor frame $\{\theta_A\}$, the spinor heat kernel $K(x, x', t)$ on S^N is a $2^{[N/2]} \times 2^{[N/2]}$ matrix and satisfies the heat equation for $\nabla\!\!\!/^2$

$$(-(\partial/\partial t) + \nabla\!\!\!/^2_x) K(x, x', t) = 0, \tag{3.1}$$

with the initial condition

$$\lim_{t\to 0} \int_{S^N} K(x, x', t)\psi(x')\,\mathrm{d}x' = \psi(x). \tag{3.2}$$

We first consider the case with N odd. The heat kernel is given by the familiar mode expansion

$$K(\theta', \Omega', \theta, \Omega, t) = \sum_{n,l,m} \big[\psi_{-nlm}(\theta', \Omega') \otimes \psi_{-nlm}(\theta, \Omega)^* + \psi_{+nlm}(\theta', \Omega') \otimes \psi_{+nlm}(\theta, \Omega)^*\big]\, \mathrm{e}^{-t(n+N/2)^2}, \tag{3.3}$$

where $\psi_{\pm nlm}(\theta, \Omega)$ is given by (2.35). In order to simplify this expression we let $\Omega = \Omega'$, i.e. we assume that the two points $x = (\theta, \Omega)$ and $x' = (\theta', \Omega')$ lie on the same "meridian".

Then we take the limit $\theta' \to 0$. We note that only the terms with $l = 0$ remain nonzero in this limit. Using (2.35) with $l = 0$ we find

$$\lim_{\theta' \to 0} K(\theta', \Omega, \theta, \Omega, t) = \mathbf{1} f_N(\theta, t), \tag{3.4}$$

where

$$f_N(\theta, t) = \frac{1}{\Omega_N} \sum_{n=0}^{\infty} d_n \phi_n(\theta) \mathrm{e}^{-t(n+N/2)^2}. \tag{3.5}$$

Here the $\phi_n(\theta)$ are the functions $\phi_{n0}(\theta)$ normalized by $\phi_n(0) = 1$,

$$\phi_n(\theta) = \frac{\phi_{n0}(\theta)}{\phi_{n0}(0)} = \frac{n!\,\Gamma(\frac{1}{2}N)}{\Gamma(n+\frac{1}{2}N)} \cos \tfrac{1}{2}\theta \; P_n^{((N/2)-1,\, N/2)}(\cos\theta), \tag{3.6}$$

and d_n are the degeneracies of $\nabla\!\!\!\!/^2$ on S^N without the spin factor $2^{[N/2]}$:

$$d_n = \frac{2\,(n+N-1)!}{n!\,(N-1)!}. \tag{3.7}$$

Eq. (3.4) agrees with the known result (see, e.g., [3]).

The spinor heat kernel $K(x', x, t)$ for arbitrary pair of points (x', x) can be written using its invariance property as

$$K(x', x, t) = U(x', x) f_N(d(x', x), t), \tag{3.8}$$

where $d(x', x)$ denotes the geodesic distance between x and x', and where $U(x', x)$ is the spinor parallel propagator from x to x'. We have used the fact that $U(x', x) = \mathbf{1}$ if x and x' lie on the same "meridian".

The case with N even can be treated in the same way. We find that the heat kernel takes the same form (3.8).

3.2. *The noncompact case*

By using the (continuous) mode expansion of the heat kernel and going through the same procedure as in the case of S^N, we find for arbitrary points $x, x' \in H^N$ with arbitrary N

$$K(x', x, t) = U(x', x) \hat{f}_N(d(x', x), t), \tag{3.9}$$

$$\hat{f}_N(y, t) = \frac{2^{N-3}\Gamma(\frac{1}{2}N)}{\pi^{(N/2)+1}} \int_0^{+\infty} \phi_\lambda(y) \mathrm{e}^{-t\lambda^2} \mu(\lambda)\, \mathrm{d}\lambda, \tag{3.10}$$

where $\phi_\lambda = \phi_{\lambda 0}$, $\mu(\lambda)$ is the spinor spectral function (2.47), and U is the parallel propagator. This result agrees with the literature (see, e.g., Ref. [3]).

4. Group theoretic derivation of the heat kernel

In this section we give a group-theoretic derivation of the spectrum, the spherical eigenfunctions, and the heat kernel of the iterated Dirac operator on S^N and H^N. We use basic facts about harmonic analysis for homogeneous vector bundles over compact symmetric spaces U/K (see, e.g., [16, Ch. 5]).

4.1. Spinors on S^N

In general, the vector bundle on U/K associated with the principal bundle $U(U/K, K)$ by a representation τ of K (see [12, Vol. I, p.55]) coincides with the homogeneous vector bundle E^τ on U/K defined by τ (see [16, Section 5.2]). Therefore we can consider (Dirac) spinor fields on $S^N = Spin(N+1)/Spin(N)$ as cross sections of the homogeneous vector bundle E^τ, where for N odd τ is the (unique) fundamental spinor representation of $Spin(N)$ (of dimension $2^{(N-1)/2}$), and for N even $\tau = \tau_+ \oplus \tau_-$, where $\tau_\pm$ are the two fundamental spinor representations of $Spin(N)$ (of dimension $2^{(N/2)-1}$ each). We let $U = Spin(N+1)$, $K = Spin(N)$ in this subsection unless specified otherwise. Let $\mathcal{U}$ and $\mathcal{K}$ be the corresponding Lie algebras. Then we have a direct sum decomposition $\mathcal{U} = \mathcal{K} \oplus \mathcal{K}^\perp$, where $\mathcal{K}^\perp$ is the orthogonal complement of $\mathcal{K}$ with respect to the Killing form. Let $x_0 = eK \in U/K$. We define the local cross section of $U(U/K, K)$, $\sigma : U/K \backslash (\text{antipodal point of } x_0) \to U$, in the following manner. Let Exp and exp denote the exponential mappings on U/K and U, respectively. Then as in any symmetric space (see [10]) $Exp\, X = \pi(exp\, X) = exp(X)x_0$, $X \in \mathcal{K}^\perp$, where $\pi : U \to U/K$ is the projection map. Hence we can choose

$$\sigma(Exp\, X) = exp\, X, \quad X \in \mathcal{P}_0, \tag{4.1}$$

$\mathcal{P}_0 = \{X \in \mathcal{K}^\perp \mid \langle X, X\rangle < \pi\}$, where $\langle\, ,\rangle = -cB$, B the Killing form on $\mathcal{U}$, and $c > 0$ is a normalization constant determined by requiring that the radius of S^N be 1. We shall use this cross section in the following.

Case 1. *N even*. The sets $\hat{U}(\tau_\pm)$ of (the equivalence classes of) the irreducible representations (irreps) of $Spin(N+1)$ that contain $\tau_\pm$ are (see, e.g., [1])

$$\hat{U}(\tau_+) = \hat{U}(\tau_-) = \{\lambda_n \equiv (n + \tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}), n = 0, 1, \ldots\}. \tag{4.2}$$

The multiplicity of $\tau_\pm$ in $\lambda_n|_K$ is 1 for all n. The iterated Dirac operator and the spinor Laplacian $L = \sum_{a=1}^N \nabla^a \nabla_a$ on a manifold M are related in general by Lichnerowicz formula

$$\nabla\!\!\!\!/^2 = L - \tfrac{1}{4}R, \tag{4.3}$$

where R is the curvature scalar of M. Then the eigenvalues $\lambda_{n,N}^2$ of $-\nabla\!\!\!\!/^2$ on S^N are given in terms of the eigenvalues ω_n of $-L$ by

$$\lambda_{n,N}^2 = \omega_n + \tfrac{1}{4}N(N-1). \tag{4.4}$$

Now using the general formula for the eigenvalues of $-L$ acting on $L^2(U/K, E^\tau)$ (see Ref. [2] for a proof) we have for E^{τ_+} ($C_2(\mu)$ denotes the Casimir number of the irrep μ)

$$\omega_n = \omega_{\lambda_n} = C_2(\lambda_n) - C_2(\tau_+). \tag{4.5}$$

(We note here that the Laplacian defined in [16, Section 5.5] has eigenvalues $-C_2(\lambda_n)$). Eq. (4.5) follows from the relation $L_{U/K} = \Omega_U - \Omega_K$, where $\Omega_U = \sum_i T_i^2$ and $\Omega_K = \sum_i X_i^2$ are the Casimir elements of U and K ($\{T_i\}$ and $\{X_i\}$ are orthonormal bases of $\mathcal{U}$ and $\mathcal{K}$, respectively). For E^{τ_-} we have the same eigenvalues since $C_2(\tau_+) = C_2(\tau_-)$. The Casimir number of an irrep with highest weight λ is given by Freudenthal's formula:

$$C_2(\lambda) = (\lambda + \rho)^2 - \rho^2 = \lambda \cdot (\lambda + 2\rho), \tag{4.6}$$

where ρ is half the sum of the positive roots of the group. The dimensions are given by the Weyl formula

$$d_\lambda = \prod_{\alpha > 0} \frac{\alpha \cdot (\lambda + \rho)}{\alpha \cdot \rho}, \tag{4.7}$$

where the product is over the positive roots α of the group. Using these (combined with (4.4) and (4.5)), we can readily verify the formulas for the eigenvalues and degeneracies of $\nabla\!\!\!\!/^2$ obtained in Section 2.

If τ is any irrep of K, the heat kernel $K(x, y, t)$ of the Laplacian L_x on E^τ is in general an element of $Hom(E_y, E_x) \simeq E_x \otimes E_y^*$ for each $t \in \mathbb{R}^+$ and $x, y \in U/K$, and satisfies

$$(-(\partial/\partial t) + L_x)\, K(x, y, t) = 0, \tag{4.8}$$

with the initial condition

$$\lim_{t \to 0^+} \int_{U/K} K(x, y, t)\psi(y)\, \mathrm{d}y = \psi(x), \tag{4.9}$$

for any continuous section $\psi \in \Gamma(E^\tau)$. Let $U^\lambda(u)^i{}_j$ ($u \in U$, $i, j = 1, \ldots, d_\lambda$) be the representation matrix for the irrep $\lambda \in \hat{U}(\tau)$ in a given orthonormal basis of V_λ. For simplicity suppose that the representation τ appears only once in $\lambda|_K$, and choose the basis $\{\mathbf{e}_j\}$ of V_λ such that the vectors $\{\mathbf{e}_a\}_{a=1,\ldots,d_\tau}$ span V_τ (the subspace of vectors of V_λ which transform under K according to τ). The matrix-valued function on U given by $f^a_{\lambda j} : u \to U^\lambda(u^{-1})^a{}_j$ satisfies $f^a_{\lambda j}(uk) = \tau(k^{-1})^a{}_b f^b_{\lambda j}(u)$ for $k \in K$. Thus for each $j = 1, \ldots, d_\lambda$ $f^a_{\lambda j}$ defines a cross section of E^τ. Let the eigenvalue of L_x acting on $f^a_{\lambda j}$ be $-\omega_\lambda$. Then, the mode expansion of the matrix heat kernel in the local basis $\theta_a(x) = \sigma(x)\mathbf{e}_a$ of $\Gamma(E^\tau)$ can be obtained in general using the Frobenius Reciprocity Theorem as

$$K(x, y, t)^a{}_b = \frac{1}{d_\tau V_{U/K}} \sum_{\lambda \in \hat{U}(\tau)} d_\lambda\, U^\lambda(\sigma^{-1}(x)\sigma(y))^a{}_b\, \mathrm{e}^{-t\omega_\lambda}, \quad a, b = 1, \ldots, d_\tau, \tag{4.10}$$

where $\sigma^{-1}(x) \equiv (\sigma(x))^{-1}$ and $V_{U/K}$ is the volume of U/K. One can express this in a coordinate invariant manner as

$$K(x,y,t)=\frac{1}{d_\tau V_{U/K}}\sum_{\lambda\in\hat{U}(\tau)} d_\lambda U(x,x_0)P_\tau U^\lambda(\sigma^{-1}(x)\sigma(y))P_\tau U(x_0,y)\mathrm{e}^{-t\omega_\lambda}, \tag{4.11}$$

where P_τ is the projector of V_λ onto V_τ, and $U(x,y)$ is the parallel propagator from E_y, the fibre at y, to E_x along the shortest geodesic between them. The U-invariance of the heat kernel allows one to let $x=x_0$ without loss of generality. Now, letting $U=Spin(N+1)$ and $K=Spin(N)$, and using (4.3), we find that the heat kernel of $\nabla\!\!\!\!/^2$ (with one point at the origin) is the direct sum $K=K^+\oplus K^-$, where the heat kernels $K^\pm$ on $E^{\tau_\pm}$ are given by

$$K^\pm(x,t)\equiv K^\pm(x_0,x,t)=\frac{1}{\Omega_N d_{\tau_\pm}}\sum_{n=0}^{\infty} d_{\lambda_n}\Phi^n_\pm(\sigma(x))U(x_0,x)\mathrm{e}^{-t\lambda^2_{n,N}}. \tag{4.12}$$

We have introduced the τ_+ (τ_-)-spherical functions Φ^n_+ (Φ^n_-), i.e. the linear operators in V_{τ_+} (V_{τ_-}) defined by

$$\Phi^n_+(u)=P_{\tau_+}U^{\lambda_n}(u)P_{\tau_+},\quad u\in U, \tag{4.13}$$

and a similar relation for Φ^n_-, where P_{τ_+} and P_{τ_-} are the projectors of V_{λ_n} onto the subspaces where $\lambda_n|_K$ is equivalent to τ_+ and τ_-, respectively.

To determine Φ^n_+, we first note that

$$\Phi^n_+(k_1uk_2)=\tau_+(k_1)\Phi^n_+(u)\tau_+(k_2),\quad u\in U,\ k_1,k_2\in K. \tag{4.14}$$

In view of the polar decomposition of U (see [10, Theorem 6.7, p.249 and Theorem 8.6, p.323]) it is enough to calculate the restrictions $\Phi^n_+(a)$, $a\in A=exp\,\mathcal{A}$, where $\mathcal{A}$ is any maximal abelian subspace of $\mathcal{K}^\perp$. In our case A is a one-dimensional group – thus $A\simeq S^1$. It is well known that M, the centralizer of A in K, may be identified with $Spin(N-1)$. From the relation $am=ma$ ($a\in A$, $m\in M$) and from (4.14) we find that the operators $\Phi^n_+(a)$, $a\in A$, commute with all the operators of the representation $\tau_+|_M=\sigma$, the unique fundamental spinor representation of $Spin(N-1)$. Since σ is irreducible, it follows from Schur's lemma that $\Phi^n_+(a)$ must be proportional to the identity operator in V_{τ_+},

$$\Phi^n_+(a)=f_n(a)\mathbf{1},\quad a\in A, \tag{4.15}$$

where f_n is a scalar function on A.

Now choose $H\in\mathcal{A}$ by requiring $\alpha(H)=1$, where α is the unique positive restricted root of $\mathcal{U}$ relative to $\mathcal{A}$ [10]. The normalization of the scalar product $\langle\,,\rangle$ on $\mathcal{A}$ (induced from that on $\mathcal{K}^\perp$) is such that $\langle H,H\rangle=1$, and corresponds to our choice $V_{U/K}=\Omega_N$. For $a\in A$ we write $a=a_\theta=exp(\theta H)$, where $\theta\in\mathbb{R}$. If $d(x,x')$ denotes the geodesic distance between $x,x'\in S^N$, then

$$d(a_\theta x_0,x_0)=d(ka_\theta x_0,x_0)=\theta,\quad k\in K, \tag{4.16}$$

where x_0 is the origin (in our case the north pole). We shall now prove that

$$f_n(a_\theta) = \phi_n(\theta), \tag{4.17}$$

where $\phi_n(\theta)$ is given by (3.6).

In order to do this, we note that a second-order differential equation for $h_\lambda(\theta) \equiv U^\lambda(a_\theta)$ $(\lambda \in \hat{U})$ is given by Lemma 2 of [14] (with an analytic continuation to the compact case, i.e. with $t \to \mathrm{i}\theta$). Let Ω_U, Ω_K and Ω_M be the quadratic Casimir operators of the algebras corresponding to the groups U, K and M. Then we find that $h_\lambda(\theta)$ satisfies the following differential equation:

$$\begin{aligned}
&\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + m_\alpha \cot\theta \frac{\mathrm{d}}{\mathrm{d}\theta}\right) h_\lambda(\theta) + h_\lambda(\theta)\Omega_M \\
&\quad - \frac{1}{\sin^2\theta}\left[(\Omega_M - \Omega_K)h_\lambda(\theta) + h_\lambda(\theta)(\Omega_M - \Omega_K) + 2\cos\theta \sum_{i=1}^{N-1} X_{\alpha i} h_\lambda(\theta) X_{\alpha i}\right] \\
&\quad = \Omega_U h_\lambda(\theta) = -C_2(\lambda) h_\lambda(\theta),
\end{aligned} \tag{4.18}$$

where $\{X_{\alpha i}\}$ is an orthonormal basis of

$$\mathcal{K}_\alpha = \{X \in \mathcal{K}:\ (ad\ H)^2 X = -X\}, \tag{4.19}$$

and we write $X_{\alpha i}$ in place of $\mathrm{d}U^\lambda(X_{\alpha i})$ or $\mathrm{d}\tau_+(X_{\alpha i})$ for simplicity. Now let $\lambda \in \hat{U}(\tau_+)$, i.e. $\lambda = \lambda_n$, and let $\Phi_+^n(a_\theta) = P_{\tau_+} h_{\lambda_n}(\theta) P_{\tau_+}$. By acting with P_{τ_+} both from the left and from the right in (4.18) we obtain an equation for $\Phi_+^n(a_\theta)$. It is clear that Ω_K and Ω_M commute with P_{τ_+}. It is also easy to see that

$$\Omega_K \Phi_+^n(a_\theta) = \Phi_+^n(a_\theta)\Omega_K = -C_2(\tau_+)\Phi_+^n(a_\theta), \tag{4.20}$$

$$\Omega_M \Phi_+^n(a_\theta) = \Phi_+^n(a_\theta)\Omega_M = -C_2(\sigma)\Phi_+^n(a_\theta). \tag{4.21}$$

Furthermore, since $\Phi_+^n(a_\theta)$ is just a scalar operator, we have

$$\sum_{i=1}^{N-1} X_{\alpha i}\Phi_+^n(a_\theta)X_{\alpha i} = \sum_{i=1}^{N-1} X_{\alpha i}X_{\alpha i}\, \Phi_+^n(a_\theta) = (C_2(\sigma) - C_2(\tau_+))\Phi_+^n(a_\theta). \tag{4.22}$$

Using these relations and substituting the explicit values of $C_2(\lambda_n)$, $C_2(\tau_+)$ and $C_2(\sigma)$ in (4.18) we find that $f_n(a_\theta)$ satisfies the same second-order differential equation as $\phi_n(\theta)$ given in (3.6). Obviously $\Phi_+^n(e) = \mathbf{1}$, and f_n must satisfy $f_n(e) = 1$. Thus (4.17) is proved and $\Phi_+^n(a_\theta) = \phi_n(\theta)\mathbf{1}$. By proceeding in a similar way with τ_- we find the same expression for $\Phi_-^n(a_\theta)$, namely $\Phi_-^n(a_\theta) = \phi_n(\theta)\mathbf{1}$. Using these formulas in (4.12) we find the heat kernel $K(x,t)$ for an arbitrary point x, which agrees with Eq. (3.8).

Case 2. *N odd*. Let τ be the fundamental spinor representation $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ of $K = Spin(N)$. The branching rule for $Spin(N+1) \supset Spin(N)$ gives

$$\hat{U}(\tau) = \{\lambda_n^+,\ n = 0, 1, \ldots\} \cup \{\lambda_n^-,\ n = 0, 1, \ldots\}, \tag{4.23}$$

where λ_n^+ and λ_n^- are the spinor representations of $Spin(N+1)$, with highest weights

$$\lambda_n^\pm = (n + \tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \pm\tfrac{1}{2}). \tag{4.24}$$

One can reproduce the formulas that we obtained for the eigenvalues $\lambda^2_{n,N}$ of $-\nabla^2$ and the degeneracies using the Freudenthal and Weyl formulas.

The heat kernel of ∇^2 (with one point at the origin) takes the form

$$\begin{aligned} K(x,t) &\equiv K(x_0,x,t) \\ &= \frac{1}{\Omega_N d_\tau} \sum_{n=0}^{\infty} d_{\lambda_n^+} \left(\Phi^{n+}(\sigma(x)) + \Phi^{n-}(\sigma(x))\right) U(x_0,x) \mathrm{e}^{-t\lambda^2_{n,N}}, \end{aligned} \tag{4.25}$$

where the τ-spherical functions $\Phi^{n\pm}$ are defined by

$$\Phi^{n\pm}(u) = P_\tau U^{\lambda_n^\pm}(u) P_\tau, \quad u \in U. \tag{4.26}$$

In order to calculate $\Phi^{n\pm}$, we notice that the operators $\Phi^{n\pm}(a)$, $a \in A$, commute with all the $\tau(m)$, $m \in M \simeq Spin(N-1)$ (the centralizer of A in K). The branching rule for $Spin(N) \supset Spin(N-1)$ gives now $\tau|_M = \sigma_+ \oplus \sigma_-$, where σ_+ and σ_- are the two fundamental spinor representations of $Spin(N-1)$. Suppose we fix an orthonormal basis of V_τ adapted to the direct sum decomposition $V_\tau = V_{\sigma_+} \oplus V_{\sigma_-}$. By applying Schur's lemma we have

$$\Phi^{n\pm}(a) = f^\pm_{n+}(a)\mathbf{1}_+ \oplus f^\pm_{n-}(a)\mathbf{1}_-, \quad a \in A, \tag{4.27}$$

where $f^\pm_{n+}$ and $f^\pm_{n-}$ are scalar functions on A and $\mathbf{1}_\pm$ denote the identity operators in $V_{\sigma_\pm}$. We shall now determine these scalar functions by using the radial part of the Casimir operator.

As before, by defining $h_\lambda(\theta) = U^\lambda(a_\theta)$, we obtain (4.18) (by [14, Lemma 2]). Now let $\lambda \in \hat{U}(\tau)$, i.e. $\lambda = \lambda_n^+$ or λ_n^-. By acting with P_τ in (4.18) both from the left and from the right we obtain an equation for $\Phi^{n+}(a_\theta)$ and $\Phi^{n-}(a_\theta)$. In this case, however, since $\Phi^{n\pm}(a_\theta)$ are not scalar operators, Eq. (4.22) is no longer valid. Let $\mathbf{diag}(p,q)$ denote the operator $p\mathbf{1}_+ \oplus q\mathbf{1}_-$ in V_τ, where $p,q \in \mathbb{C}$. Then it is not difficult to show, using the explicit formula for $X_{\alpha i}$ in the irrep τ, that

$$\sum_{i=1}^{N-1} X_{\alpha i}\, \mathbf{diag}(p,q) X_{\alpha i} = -\frac{N-1}{4}\mathbf{diag}(q,p). \tag{4.28}$$

Using (4.28) in (4.18) we get the following set of coupled equations for the functions $f^+_{n+}(a_\theta)$ and $f^+_{n-}(a_\theta)$:

$$\begin{aligned} &\left[\partial_\theta^2 + (N-1)\cot\theta\,\partial_\theta - C_2(\sigma_+)\right] f^+_{n\pm} - \frac{N-1}{2\sin^2\theta} f^+_{n\pm} + \frac{(N-1)\cos\theta}{2\sin^2\theta} f^+_{n\mp} \\ &\quad = -C_2(\lambda_n^+) f^+_{n\pm}. \end{aligned} \tag{4.29}$$

The same set of equations (with $f^+_{n\pm} \to f^-_{n\pm}$) is obtained for the irreps $\lambda = \lambda_n^-$.

By taking the sum and the difference of the two equations in (4.29), it is then easy to see that the function $f^+_{n+}(a_\theta) + f^+_{n-}(a_\theta)$ satisfies the same differential equation (2.15) as $\phi_n(\theta)$ (given by (3.6)), and the function $f^+_{n+}(a_\theta) - f^+_{n-}(a_\theta)$ satisfies the same equation as the function $\psi_n(\theta)$ given by

$$\psi_n(\theta) = \frac{\psi_{n0}(\theta)}{\phi_{n0}(0)} = \frac{n!\,\Gamma(\frac{1}{2}N)}{\Gamma(n+\frac{1}{2}N)} \sin\tfrac{1}{2}\theta\, P_n^{(N/2,\,(N/2)-1)}(\cos\theta), \tag{4.30}$$

in the notations of Section 2. [The equation for ψ_n is the same as (2.15) (for $l = 0$) with the replacement $\theta \to \pi - \theta$.]

Thus $f^{+}_{n+}(a_\theta) + f^{+}_{n-}(a_\theta) \propto \phi_n(\theta)$, and since $\phi_n(0) = f^{+}_{n\pm}(e) = 1$ we have

$$f^{+}_{n+}(a_\theta) + f^{+}_{n-}(a_\theta) = 2\phi_n(\theta). \tag{4.31}$$

Similarly, we have

$$f^{+}_{n+}(a_\theta) - f^{+}_{n-}(a_\theta) = 2\mathrm{i}\psi_n(\theta), \tag{4.32}$$

where $f^{+}_{n+}(a_\pi) = -f^{+}_{n-}(a_\pi) = (-1)^n\mathrm{i}$ has been used. (The equation $\Phi^{n\pm}(a_\pi) = \pm\mathrm{i}(-1)^n\Gamma^N$ can easily be proved.) Thus,

$$f^{+}_{n\pm}(a_\theta) = \phi_n(\theta) \pm \mathrm{i}\psi_n(\theta). \tag{4.33}$$

For the functions $f^{-}_{n\pm}$ we obtain similarly

$$f^{-}_{n\pm}(a_\theta) = \phi_n(\theta) \mp \mathrm{i}\psi_n(\theta). \tag{4.34}$$

Using (4.33)–(4.34) in (4.27) and then in (4.25) we reproduce the heat kernel $K(x, x', t)$ given by (3.8).

4.2. Spinors on H^N

Let us first review some general results. Let G be a noncompact semisimple Lie group with finite center, K a maximal compact subgroup, and G/K the associated Riemannian symmetric space of the noncompact type. Harmonic analysis on G/K is well understood in the case of scalars. For vector bundles L^2-harmonic analysis on G/K may be reduced to L^2-harmonic analysis on the group G, by letting $L^2(G/K, E^\tau)$ ($\tau \in \hat{K}$) sit in $L^2(G)$ in a natural way. The important point here is that the heat kernel of the Laplacian acting on $L^2(G/K, E^\tau)$ may be expanded in terms of the τ-spherical functions in a way which is similar to (4.12) with $\sum_{\lambda\in\hat{U}(\tau)} d_\lambda \to \int_{\hat{G}(\tau)} \mathrm{d}\mu(\lambda)$, i.e.

$$K(x,t) \equiv K(x_0, x, t) = \frac{1}{d_\tau} \int\limits_{\hat{G}(\tau)} \Phi^\lambda_\tau(\sigma(x)) U(x_0, x)\, \mathrm{e}^{-t\omega_\lambda}\, \mathrm{d}\mu(\lambda), \tag{4.35}$$

where $\mathrm{d}\mu(\lambda)$ is the Plancherel measure (see, e.g., [11]). The notations here are as follows. The τ-spherical functions are the (operator-valued) functions $g \to \Phi^\lambda_\tau(g)$ on G defined by

$$\Phi^\lambda_\tau(g) = P_\tau U^\lambda(g) P_\tau, \quad g \in G, \tag{4.36}$$

where $U^\lambda(g)$ denote the operators of the representation $\lambda \in \hat{G}(\tau)$ (this set has the same meaning as for the compact group U) in a Hilbert space H_λ, and P_τ is the projector of H_λ onto V_τ, the subspace of vectors of H_λ which transform under K according to τ. (Again, we assume that the irrep τ appear only once in λ for simplicity.) As in the compact case $\sigma(Exp\, X) = exp\, X$ ($X \in \mathcal{K}^{\perp}$), and $U(x_0, x)$ denotes the (vector bundle) parallel transport operator from x to x_0 along the geodesic between them. The vector bundle Laplacian $L_{G/K}$ takes the form

$$L_{G/K} = \Omega_G + \Omega_K, \tag{4.37}$$

where Ω_G and Ω_K are the quadratic Casimir operators of $\mathcal{G}$ and $\mathcal{K}$. The plus sign in front of Ω_K is due to the fact that the Killing form has opposite signature on $\mathcal{K}$ and $\mathcal{K}^{\perp}$. Denoting the Casimir value of λ by $-C_2(\lambda)$, we find the eigenvalues $-\omega_\lambda$ of $L_{G/K}$ from (4.37) as

$$-\omega_\lambda = -C_2(\lambda) - C_2(\tau). \tag{4.38}$$

There is a well-known duality between the noncompact and the compact symmetric spaces. Consider the subspace $\mathcal{U} = \mathcal{K} \oplus \mathrm{i}\mathcal{K}^{\perp}$ of the complexification $\mathcal{G}^{\mathbb{C}}$ of $\mathcal{G}$, where $\mathrm{i} = \sqrt{-1}$. The Lie algebra $\mathcal{U}$ is called a *compact form* of $\mathcal{G}$ (cf. [11, p.114]). It is a compact Lie algebra since the Killing form is negative definite on $\mathcal{U}$. Let U denote a simply connected compact Lie group with Lie algebra $\mathcal{U}$. Then U/K is the compact symmetric space which is *dual* to G/K [10]. It is clear that the radial parts of the Casimir operators on G and U (acting on τ-spherical functions) will be related by analytic continuation through $h \to \mathrm{i}h$, for $h \in \mathcal{A}$. Now, let $G = Spin(N,1)$, $K = Spin(N)$, $G/K = H^N$.

Case 1. *N even.* Let $\tau = \tau_+ \oplus \tau_-$. (We use the same notations as in Section 4.1.) From the branching rule for $Spin(N,1) \supset Spin(N)$ we find that no discrete series contain τ (see [6]). Let $U_{(\mathrm{i}\lambda,\sigma)}$ denote the unitary principal series representation labelled by $\lambda \in \mathbb{R}^+$ and $\sigma \in \hat{M}$, where $M = Spin(N-1)$ is the centralizer of A in K and $G = KAN$ is an Iwasawa decomposition of G [16]. This is the representation of G induced by the representation D of a minimal parabolic subgroup MAN, where

$$D(man) = \sigma(m)\mathrm{e}^{\mathrm{i}\lambda(\log a)}. \tag{4.39}$$

By Frobenius Reciprocity $U_{(\mathrm{i}\lambda,\sigma)}|_K$ contains τ if and only if $\tau|_M$ contains σ. Thus we find that the unitary principal series containing both τ_+ and τ_- are the $U_{(\mathrm{i}\lambda,\sigma)}$ with $\sigma = (\frac{1}{2}, \ldots, \frac{1}{2})$, the fundamental spinor representation of M. The Plancherel measure for this principal series is (see [6])

$$\mathrm{d}\mu(U_{(\mathrm{i}\lambda,\sigma)}) = \frac{2^{N-2}}{\pi\,\Omega_{N-1}} d_\sigma |C(\lambda)|^{-2}\,\mathrm{d}\lambda, \tag{4.40}$$

$$C(\lambda) = \frac{2^{N-2}\Gamma(\frac{1}{2}N)}{\sqrt{\pi}} \frac{\Gamma(\mathrm{i}\lambda + \frac{1}{2})}{\Gamma(\mathrm{i}\lambda + \frac{1}{2}N)}, \tag{4.41}$$

where $d_\sigma = 2^{(N/2)-1}$ is the dimension of σ and Ω_{N-1} is the volume of S^{N-1} (cf. (2.27)). The radial part of the Casimir operator acting on the restrictions $\Phi^{\lambda}_{\pm}(a_y)$, $a_y = exp(yH) \in A$, may be obtained from (4.18) by letting $\theta \to \mathrm{i}y$ and applying P_{τ_+} (or P_{τ_-}) both from the left and from the right. Again we find, using Schur's lemma, $\Phi^{\lambda}_{+}(a_y) = \phi_\lambda(y)\mathbf{1}$, where ϕ_λ

is a scalar function on A. The Casimir value $-C_2(\lambda)$ for $U_{(i\lambda,\sigma)}$ may be easily calculated and is given by

$$-C_2(\lambda) = -\lambda^2 - \tfrac{1}{8}N(N-1). \tag{4.42}$$

Then from (4.3) we find that the eigenvalues of ∇^2 are given by $-\lambda^2$, $\lambda \in \mathbb{R}$. The resulting differential equation satisfied by the scalar function $\phi_\lambda(y)$ is obtained from that for $f_n(a_\theta)$ in the compact case by letting $\theta \to \mathrm{i}y$ and $n \to -\mathrm{i}\lambda - \frac{1}{2}N$. We find the solution to be

$$\phi_\lambda(y) = \cosh \tfrac{1}{2}y\, F(\mathrm{i}\lambda + \tfrac{1}{2}N,\ -\mathrm{i}\lambda + \tfrac{1}{2}N, \tfrac{1}{2}N,\ -\sinh^2 \tfrac{1}{2}y). \tag{4.43}$$

Similarly we find $\Phi^\lambda_-(a_y) = \phi_\lambda(y)\mathbf{1}$. Using the above formulas in (4.35) we find for the heat kernel $K = K^+ \oplus K^-$,

$$K^+(x,t) \oplus K^-(x,t) = U(x_0,x)\,\frac{2^{N-3}\Gamma(\frac{1}{2}N)}{\pi^{(N/2)+1}} \int_0^{+\infty} \phi_\lambda(y)\mathrm{e}^{-t\lambda^2}|C(\lambda)|^{-2}\,\mathrm{d}\lambda, \tag{4.44}$$

where $y = d(x_0,x)$. This agrees with (3.9) and (3.10).

Case 2: *N odd*. Let τ be the irrep $(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$ of K. By Frobenius Reciprocity we find that the unitary principal series representations containing τ are $U_{(\mathrm{i}\lambda,\sigma_+)}$ and $U_{(\mathrm{i}\lambda,\sigma_-)}$, where $\sigma_\pm = (\frac{1}{2},\ldots,\frac{1}{2},\ \pm\frac{1}{2})$ are the two fundamental spinor representations of $M = Spin(N-1)$. The Plancherel measure $\mathrm{d}\mu(U_{(\mathrm{i}\lambda,\sigma_+)}) = \mathrm{d}\mu(U_{(\mathrm{i}\lambda,\sigma_-)})$ is given again by (4.40) (with $d_\sigma \to d_{\sigma_+} = 2^{(N-3)/2}$) and (4.41).

Since $\tau|_M = \sigma_+ \oplus \sigma_-$, the τ-spherical functions $\Phi^{\lambda\pm}(g) \equiv P_\tau U_{(\mathrm{i}\lambda,\sigma_\pm)}(g)P_\tau$ at $g = a_y = exp(yH) \in A$ are given by (cf. (4.27))

$$\Phi^{\lambda\pm}(a_y) = \mathbf{diag}(f^\pm_{\lambda+}(y), f^\pm_{\lambda-}(y)). \tag{4.45}$$

The scalar functions $f^\pm_{\lambda+}$ and $f^\pm_{\lambda-}$ may be obtained from $f^\pm_{n+}$ and $f^\pm_{n-}$, respectively, in (4.33) and (4.34) by letting $n \to -\mathrm{i}\lambda - \frac{1}{2}N$ and changing the argument $\theta \to \mathrm{i}y$ because the differential equations are related in this manner. Thus

$$f^+_{\lambda\pm} = \phi_\lambda \pm \mathrm{i}\psi_\lambda = f^-_{\lambda\mp}, \tag{4.46}$$

where $\phi_\lambda(y)$ is given by (4.43) and

$$\psi_\lambda(y) = (2\lambda/N)\sinh \tfrac{1}{2}y\, F(\mathrm{i}\lambda + \tfrac{1}{2}N,\ -\mathrm{i}\lambda + \tfrac{1}{2}N, \tfrac{1}{2}N+1,\ -\sinh^2 \tfrac{1}{2}y). \tag{4.47}$$

Using the above equations in (4.35) we find that the heat kernel $K(x,t)$ is given by the right-hand side of (4.44), again in agreement with our results in Section 3.

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